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$\mathbb{Z}_2 \times \mathbb{Z}_2$ lattice as a Connes–Lott-quantum group model

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Abstract

We apply quantum group methods for noncommutative geometry to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ lattice to obtain a natural Dirac operator on this discrete space. This then leads to an interpretation of the Higgs fields as the discrete part of space–time in the Connes–Lott formalism for elementary particle Lagrangians. The model provides a setting where both the quantum groups and the Connes approach to noncommutative geometry can be usefully combined, with some of Connes' axioms, notably the first-order condition, replaced by algebraic methods based on the group structure. The noncommutative geometry has nontrivial cohomology and moduli of flat connections, both of which we compute. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [1,2], Connes and Lott proposed a framework for the standard model in elementary particle physics based on a discrete and typically noncommutative part adjoined to conventional space–time. Fields on this composite space–time appear as multiplets of fields on ordinary space–time and, for the right choice of discrete part, one obtains exactly the standard model of particle physics. The Dirac operator on the discrete part encodes the masses of fermions on usual space–time. This approach 'packages' the standard model into

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an elegant framework where also the Higgs field arises naturally. However, most of the parameters of the standard model are still left undetermined because in the Connes approach to noncommutative geometry almost any self-adjoint operator $\partial\!\!\!/$ can be taken on the discrete part of space–time in the role of Dirac.

Meanwhile coming from quantum groups is a 'constructive' approach to noncommutative geometry which includes also finite groups and other discrete spaces. In this approach, because of the existence of q-deformed examples one keeps 'eye contact' with conventional geometric ideas and thereby builds up the different layers of (noncommutative) geometry up to and including, in recent work [3], the Dirac operator. In other words when the Connes–Lott formalism and the quantum groups formalism are combined one has natural 'geometric' criteria for the choice of Dirac operator on the discrete space–time which translates directly into predictions in elementary particle physics.

In this paper, we develop a nontrivial model for which these two approaches can be combined in this way, and explore fully both approaches for this model. The model has 'discrete part' $\mathbb{Z}_2 \times \mathbb{Z}_2$ which has a commutative coordinate algebra but which we equip with noncommutative differentials coming naturally from the quantum groups approach (a bicovariant differential calculus in the sense of Woronowicz [4]). The model is too simple to lead to exactly the standard model (for this one wants the noncommutative algebra $\mathbb{C}\oplus\mathbb{H}\oplus M_3$) but it exhibits many of the same features. Moreover, the model is of independent interest as a discrete (lattice) model of space–time useful in a variety of other contexts, e.g. potentially for QCD.

In Section 2 we explore the model using Connes' formalism [1,2]. Thus, starting with the bicovariant differential calculus suggested by quantum group methods, we take the natural two-dimensional Dirac operator and apply the method of Connes to induce an entire exterior algebra, Hodge $*$ and other constructions on this discrete 2D 'space–time' (not to be confused with conventional space–time of course but thought of in that way). We find a Higgs-effect and aspects of symmetry breaking on this discrete space–time. Following Connes, we work very explicitly with 1-forms and 2-forms, etc. as certain concrete matrices. The higher forms are not so easily computed by these methods, however.

In Section 3, we construct this exterior algebra, etc. induced by $\partial\!\!\!/$ from a more algebraic point of view using quantum group methods. Here the exterior algebra is obtained as a quotient of the universal differential calculus by generators and relations, and not concretely given by particular matrices. We show how many of the computations in the Connes–Lott model building kit can be done more in line with classical constructions using these algebraic quantum group methods. Using these methods we are then able to take the computations of Section 2 much further. We fully compute the exterior algebra, its quantum de Rham cohomology and its moduli of zero curvature gauge fields, all of which turn out to be nontrivial. We note that quantum group methods for the noncommutative geometry on finite groups have recently been developed in some generality [3,5], including gravity and a first contact with Connes' method which we use now (an analysis of the 2-forms). See also [6] where the cohomology and gauge theory for the permutation group S_3 is recently computed. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ model can be viewed as another nontrivial noncommutative geometry in this family. The use of noncommutative geometry for discrete space–time itself originates in the bilocal nature of finite difference differentials, and is quite fundamental.

In Section 4, we return to the physics by combining this discrete 2D space–time with conventional space–time to pull out the resulting fairly straightforward model of particle physics and some predictions ensuing form our particular 'geometrical' choice of $\partial\!\!\!/$. In Section 5, we look at a further chapter of Connes method [7,8], namely the spectral action and gravity, automorphisms and spin automorphisms. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ model is again simple enough that all aspects are computable explicitly.

Finally, in Section 6 we conclude with some comments on the general lattice $(\mathbb{Z}_m)^n$. Using again our algebraic quantum group methods we show that similar features hold for higher dimensional $(\mathbb{Z}_2)^n$ but that on the other hand, for $m > 2$ the nontrivial features of the model such as the Higgs potential disappear, i.e. are a very specific to the use of $\mathbb{Z}_2$.

2. The 2×2 lattice à la Connes–Lott

In this section, we apply the Connes–Lott model building kit [1] to a 2×2 lattice described by the associative, unital star algebra

$$\mathcal{A} = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2] \ni f(x, y), \quad x, y = 0, 1 \mod 2. \tag{1}$$

We define right translations in x and y directions by

$$(R_x f)(x, y) := f(x + 1, y), \qquad (R_y f)(x, y) := f(x, y + 1), \tag{2}$$

and the partial derivatives by

$$\partial_x := R_x - 1, \qquad \partial_y := R_y - 1. \tag{3}$$

The following relations will be useful:

$$(R_x)^2 = 1, \qquad (R_y)^2 = 1, \qquad R_x R_y = R_y R_x, \tag{4}$$

$$(\partial_x)^2 = -2\partial_x, \qquad (\partial_y)^2 = -2\partial_y, \qquad \partial_x \partial_y = \partial_y \partial_x, \tag{5}$$

and the Leibniz rule

$$\partial_x (fg) = (\partial_x f)g + (R_x f)\partial_x g = (\partial_x f)R_x g + f\partial_x g. \tag{6}$$

We define the Hilbert space of spinors

$$\mathcal{H} := \mathcal{A} \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \psi_R, \psi_L \in \mathcal{A} \tag{7}$$

with scalar product

$$(\psi, \tilde{\psi}) := \sum_{x,y=0}^{1} [\bar{\psi}_R(x, y)\tilde{\psi}_R(x, y) + \bar{\psi}_L(x, y)\tilde{\psi}_L(x, y)], \tag{8}$$

and the faithful representation ρ of the algebra $\mathcal{A}$ on $\mathcal{H}$ by pointwise multiplication

$$(\rho(f)\psi)(x, y) := f(x, y)\psi(x, y). \tag{9}$$

We will need the relation

$$(R_x \otimes 1_2)\rho(f) = \rho(R_x f)(R_x \otimes 1_2). \tag{10}$$

The third input item is the Dirac operator that we take to be the lattice Dirac operator,

$$\not{\partial} := \partial_x \otimes \gamma^x + \partial_y \otimes \gamma^y \tag{11}$$

with the Hermitian Pauli matrices

$$\gamma^x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^y := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

They satisfy

$$(\gamma^x)^2 = (\gamma^y)^2 = 1_2, \qquad \gamma^x\gamma^y = -\gamma^y\gamma^x = \mathrm{i}\gamma^3. \tag{13}$$

Note that this Dirac operator is self-adjoint without an imaginary i in front. Note also that this Dirac operator like any lattice Dirac operator cannot satisfy Connes' [2] first-order condition [11]. In the commutative case of a Riemannian spin manifold, this algebraic condition reduces to the property that the Dirac operator is a first-order differential operator. Even without the first-order condition we can use these three items, the algebra $\mathcal{A}$, its representation on $\mathcal{H}$ and the Dirac operator $\not{\partial}$ to construct a Connes–Lott model [1].

The first step involves an auxiliary differential algebra $\Omega_{\mathrm{univ}}(\mathcal{A})$, the universal exterior algebra of $\mathcal{A}$:

$$\Omega^0_{\mathrm{univ}}(\mathcal{A}) := \mathcal{A}, \tag{14}$$

$\Omega^1_{\mathrm{univ}}(\mathcal{A})$ is spanned over $\mathcal{A}$ by symbols $\mathrm{d}a$, $a \in \mathcal{A}$ with relations

$$\mathrm{d}1 = 0, \qquad \mathrm{d}(ab) = (\mathrm{d}a)b + a\,\mathrm{d}b. \tag{15}$$

Therefore it consists of finite sums of terms of the form $a_0\,\mathrm{d}a_1$,

$$\Omega^1_{\mathrm{univ}}(\mathcal{A}) = \left\{ \sum_j a_0^j\,\mathrm{d}a_1^j,\ a_0^j, a_1^j \in \mathcal{A} \right\}, \tag{16}$$

and likewise for higher p,

$$\Omega^p_{\mathrm{univ}}(\mathcal{A}) = \left\{ \sum_j a_0^j\,\mathrm{d}a_1^j \cdots \mathrm{d}a_p^j,\ a_q^j \in \mathcal{A} \right\}. \tag{17}$$

The differential d is defined by $\mathrm{d}(a_0\,\mathrm{d}a_1 \cdots \mathrm{d}a_p) := \mathrm{d}a_0\,\mathrm{d}a_1 \cdots \mathrm{d}a_p$. The involution * is extended from the algebra $\mathcal{A}$ to $\Omega^1_{\mathrm{univ}}(\mathcal{A})$ by putting

$$(\mathrm{d}a)^* := -\mathrm{d}(a^*). \tag{18}$$

Some authors, including Connes, use $(\mathrm{d}a)^* := \mathrm{d}(a^*)$ which amounts to replacing d by $-$id. With the definition $(\alpha\beta)^* = \beta^*\alpha^*$ for forms α, β, the involution is extended to the whole universal exterior algebra.

The next step is to extend the representation ρ on $\mathcal{H}$ from the algebra $\mathcal{A}$ to its universal exterior algebra. This extension is the central piece of Connes' algorithm:

$$\pi : \Omega_{\text{univ}}(\mathcal{A}) \to \text{End}(\mathcal{H}), \qquad \pi(a_0\,\mathrm{d}a_1 \cdots \mathrm{d}a_p) := \rho(a_0)[\partial\!\!\!/, \rho(a_1)] \cdots [\partial\!\!\!/, \rho(a_p)]. \tag{19}$$

A straightforward calculation shows that π is in fact a representation of $\Omega_{\text{univ}}(\mathcal{A})$ as an algebra with involution, and we are tempted to define also a differential, denoted again by d, on the images $\pi(\Omega^p_{\text{univ}}(\mathcal{A}))$ in each degree by

$$\mathrm{d}\pi(\alpha) := \pi(\mathrm{d}\alpha) \quad \forall \alpha \in \Omega^p_{\text{univ}}(\mathcal{A}). \tag{20}$$

However, this definition does not make sense if there are forms $\alpha \in \Omega_{\text{univ}}(\mathcal{A})$ with $\pi(\alpha) = 0$ and $\pi(\mathrm{d}\alpha) \neq 0$. By dividing out these unpleasant forms, Connes constructs a new differential algebra $\Omega_{\partial\!\!\!/}(\mathcal{A})$, the interesting object

$$\Omega_{\partial\!\!\!/}(\mathcal{A}) := \frac{\pi(\Omega_{\text{univ}}(\mathcal{A}))}{\mathcal{J}} \tag{21}$$

with

$$\mathcal{J} := \pi(\mathrm{d} \ker \pi) =: \bigoplus_p \mathcal{J}_p \tag{22}$$

($\mathcal{J}$ for junk). On the quotient now, the differential (20) is well defined. Degree by degree we have

$$\Omega^0_{\partial\!\!\!/}(\mathcal{A}) = \rho(\mathcal{A}) \tag{23}$$

because $\mathcal{J}^0 = 0$,

$$\Omega^1_{\partial\!\!\!/}(\mathcal{A}) = \pi(\Omega^1_{\text{univ}}(\mathcal{A})) \tag{24}$$

because ρ is faithful, and in degree $p \geq 2$,

$$\Omega^p_{\partial\!\!\!/}(\mathcal{A}) = \frac{\pi(\Omega^p_{\text{univ}}(\mathcal{A}))}{\pi(\mathrm{d} \ker \pi_{p-1})}. \tag{25}$$

Here π_{p-1} denotes π restricted to degree $p-1$ forms. We remind the motivation of Connes' construction: in the continuum, $\mathcal{A}$ the algebra of differentiable functions on the 2-torus and $\partial\!\!\!/$ the genuine Dirac operator, $\Omega_{\partial\!\!\!/}(\mathcal{A})$ is de Rham's exterior algebra of differential forms.

In our lattice model all forms are explicit 8×8 matrices. For example in Eq. (28)

$$\omega_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{26}$$

with respect to the basis

$$\{\delta_{00}, \delta_{10}, \delta_{01}, \delta_{11}\} \otimes \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Let us compute the 1-forms

$$\pi(\mathrm{d}f) = [\partial\!\!\!/, \rho(f)] = \rho(\partial_x f)R_x \otimes \gamma^x + \rho(\partial_y f)R_y \otimes \gamma^y. \tag{27}$$

We denote by $\delta_{00}, \delta_{10}, \delta_{01}, \delta_{11} \in \mathcal{A}$ the four delta functions, δ_{00} for instance is one on $x = 0$, $y = 0$ and zero on the other three points. Then $\Omega^1_{\partial\!\!\!/}(\mathcal{A})$ is spanned over $\mathcal{A}$ by the two elements

$$\omega_x := \pi(\partial_x X\, \mathrm{d}X) = R_x \otimes \gamma^x, \qquad \omega_y := \pi(\partial_y Y\, \mathrm{d}Y) = R_y \otimes \gamma^y, \tag{28}$$

where we have put $X := \delta_{10} + \delta_{11}$ and $Y := \delta_{01} + \delta_{11}$. Our generators are Hermitian, $\omega_x^* = \omega_x, \omega_y^* = \omega_y$.

The 2-forms are represented by

$$\begin{aligned}\pi(\mathrm{d}f\, \mathrm{d}g) = {} & \rho(\partial_x f\, R_x \partial_x g + \partial_y f\, R_y \partial_y g) \otimes 1_2 \\ & + \rho(\partial_x f\, R_x \partial_y g - \partial_y f\, R_y \partial_x g) R_x R_y \otimes \gamma^x \gamma^y,\end{aligned} \tag{29}$$

and a straightforward calculation in a basis using δ-functions shows that in degree 2 the junk vanishes, $\mathcal{J}_2 = 0$. Therefore, we get

$$(\omega_x)^2 = (\omega_y)^2 = 1, \qquad \omega_x\omega_y = -\omega_y\omega_x, \qquad \mathrm{d}\omega_x = \mathrm{d}\omega_y = 2. \tag{30}$$

At this stage there is a first contact with gauge theories. Consider the vector space of Hermitian 1-forms $\{H \in \Omega^1_{\partial\!\!\!/}(\mathcal{A}),\, H^* = H\}$. A general element H is of the form

$$H = \rho(h_x)\omega_x + \rho(h_y)\omega_y \tag{31}$$

with

$$\begin{aligned} h_x(0,0) &= h_x(1,0)^*, \qquad h_x(1,1) = h_x(0,1)^*, \\ h_y(0,0) &= h_y(0,1)^*, \qquad h_y(1,1) = h_y(1,0)^*. \end{aligned} \tag{32}$$

These elements H are gauge potentials on the lattice. In fact, the space of gauge potentials carries an affine representation of the group of unitaries

$$U(\mathcal{A}) := \{u \in \mathcal{A}, uu^* = u^*u = 1\} =: \mathcal{G}. \tag{33}$$

In our example this group is Maxwell's local $U(1)$. In general its action is defined by

$$\begin{aligned} H^u &:= \rho(u)H\rho(u^{-1}) + \rho(u)\,\mathrm{d}(\rho(u^{-1})) = \rho(u)H\rho(u^{-1}) + \rho(u)[\partial\!\!\!/, \rho(u^{-1})] \\ &= \rho(u)[H + \partial\!\!\!/]\rho(u^{-1}) - \partial\!\!\!/. \end{aligned} \tag{34}$$

H^u is the "gauge transformed of H". As usual every gauge potential H defines a covariant derivative $\mathrm{d} + H$, covariant under the left action of $\mathcal{G}$ on $\Omega_{\partial\!\!\!/}\mathcal{A}$:

$${}^u\omega := \rho(u)\omega, \quad \omega \in \Omega_{\partial\!\!\!/}\mathcal{A}, \tag{35}$$

which means

$$(\mathrm{d} + H^u)\,{}^u\omega = {}^u[(\mathrm{d} + H)\omega]. \tag{36}$$

Also we define the curvature C of H by

$$C := \mathrm{d}H + H^2 \in \Omega^2_{\partial\!\!\!/}(\mathcal{A}). \tag{37}$$

The curvature C is a Hermitian 2-form with *homogeneous* gauge transformations

$$C^u := \mathrm{d}(H^u) + (H^u)^2 = \rho(u) C \rho(u^{-1}). \tag{38}$$

In our example, we get

$$\begin{aligned} C = {} & \rho(\partial_x h_x + \partial_y h_y + 2h_x + 2h_y + h_x R_x h_x + h_y R_y h_y)\omega_x^2 \\ & + \rho(-\partial_y h_x + \partial_x h_y + h_x R_x h_y - h_y R_y h_x)\omega_x \omega_y. \end{aligned} \tag{39}$$

In the last step, we construct the Yang–Mills action. To this end we need a scalar product on the space of 2-forms. But our forms are operators on the finite dimensional Hilbert space $\mathcal{H}$ and we have a natural scalar product.

At this point we must note that although ρ is faithful π is not, not even after dividing out the junk; and even worse, the image in $\mathrm{End}(\mathcal{H})$ does not remember its degree. (This complication does not occur in the continuum.) In our example for instance we meet the 8×8 unit matrix as 0-form $\rho(1)$ and as 2-form ω_x^2. By definition the scalar product of two forms of different degree is taken to be zero, for forms of same degree p, we define

$$(\omega, \tilde{\omega}) := \mathrm{tr}(\omega^* \tilde{\omega}), \quad \omega, \tilde{\omega} \in \Omega^p_{\partial\!\!\!/}(\mathcal{A}). \tag{40}$$

For example, $\rho(1), \omega_x, \omega_y, \omega_x\omega_y, \omega_x^2$ are orthogonal generators, all normed to $\sqrt{8}$. More generally, we have

$$\begin{aligned} & (\rho(f), \rho(\tilde{f})) = 2(f, \tilde{f}), \qquad (\rho(f)\omega_x, \rho(\tilde{f})\omega_x) = 2(f, \tilde{f}), \\ & (\rho(f)\omega_y, \rho(\tilde{f})\omega_y) = 2(f, \tilde{f}), \end{aligned} \tag{41}$$

$$(\rho(f)\omega_x\omega_y, \rho(\tilde{f})\omega_x\omega_y) = 2(f, \tilde{f}), \qquad (\rho(f)\omega_x^2, \rho(\tilde{f})\omega_x^2) = 2(f, \tilde{f}) \tag{42}$$

with

$$(f, \tilde{f}) := \sum_{x,y=0}^{1} \bar{f}(x, y)\tilde{f}(x, y). \tag{43}$$

We are now in position to define the Yang–Mills action $V_0(H) = (C, C)$. By construction it is a positive, gauge invariant polynomial of fourth order in the values of h_x and h_y. Its minimum, $H = 0$, breaks the gauge invariance. In order to compute the Yang–Mills action, we introduce a new variable [12]

$$\varphi := H + \partial\!\!\!/_{\mathcal{G}} \tag{44}$$

with

$$\partial\!\!\!/_{\mathcal{G}} := -\int_{\mathcal{G}} \pi(u^{-1}\,\mathrm{d}u)\,\mathrm{d}u = \partial\!\!\!/ - \int_{\mathcal{G}} \rho(u^{-1})\partial\!\!\!/\rho(u)\,\mathrm{d}u = \omega_x + \omega_y, \tag{45}$$

and du is the normalized Haar measure of the compact Lie group $\mathcal{G}$. We decide that the Dirac operator does not transform under gauge transformations. Then φ transforms homogeneously

$$\varphi^u = \rho(u)\varphi\rho(u^{-1}). \tag{46}$$

Let us expand the homogeneous variable as

$$\varphi = \rho(\varphi_x)\omega_x + \rho(\varphi_y)\omega_y \tag{47}$$

with $\varphi_x = h_x + 1$, $\varphi_y = h_y + 1$. Then we can rewrite the curvature as

$$C = \rho(\varphi_x R_x \varphi_x + \varphi_y R_y \varphi_y - 2)\omega_x^2 + \rho(\varphi_x R_x \varphi_y - \varphi_y R_y \varphi_x)\omega_x\omega_y, \tag{48}$$

and the Yang–Mills action can be written explicitly

$$\begin{aligned} V_0 = 2\{&[|\varphi_x(0,0)|^2 + |\varphi_y(0,0)|^2 - 2]^2 + [|\varphi_x(0,0)|^2 + |\varphi_y(1,1)|^2 - 2]^2 \\ &+ [|\varphi_x(1,1)|^2 + |\varphi_y(0,0)|^2 - 2]^2 + [|\varphi_x(1,1)|^2 + |\varphi_y(1,1)|^2 - 2]^2 \\ &+ 2|\varphi_x(0,0)\varphi_y(1,1)^* - \varphi_x(1,1)^*\varphi_y(0,0)|^2 + 2|\varphi_x(0,0)\varphi_y(0,0)^* \\ &- \varphi_x(1,1)^*\varphi_y(1,1)|^2\}. \end{aligned} \tag{49}$$

The little group of its minimum $H = 0$ or $\varphi := \partial\!\!\!/_{\mathcal{G}}$ is the group of rigid $U(1)$ transformations as in the continuous case. However, unlike in the continuous case, there is a gauge invariant point, $\varphi = 0$ or $H = -\partial\!\!\!/_{\mathcal{G}}$ which is also a local maximum of the Yang–Mills action V_0. The existence of this gauge invariant point indicates that in this model, H plays simultaneously the role of the gauge potential and the role of a Higgs scalar. The lattice Yang–Mills action is its Higgs potential.

The minima of the potential V_0 are continuously degenerate, $\varphi_x(0,0) = \varphi_x(1,1) = \sqrt{2}\sin\beta$, $\varphi_y(0,0) = \varphi_y(1,1) = \sqrt{2}\cos\beta$. All minima have little groups $U(1)$ except when β is an integer multiple of $\pi/2$. Then the little group is $U(1)^2$. Let us remark that this model is similar to Example 3.1 in [12]: its algebra is represented vectorially, but does not commute with the Dirac operator and its potential has degenerate minima with different little groups.

3. Quantum group methods for the same model

In the previous section, we have pulled the partial derivatives and Dirac operator 'out of a hat' (motivated of course by the wish to include lattice differentials). In particular, since the resulting $\partial\!\!\!/$ does not obey the first-order condition in Connes's axioms in any standard way, it is not motivated from that theory. Rather this choice of differentials comes from requiring translation invariance under the group structure of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. This is part of the 'quantum groups approach' where one builds up the different layers of noncommutative geometry based on the group or quantum group structure. This approach also has a more algebraic way of working in which we deal algebraically with the differential forms rather than concretely as matrices. In this section, we explain the construction of the Dirac operator from the quantum groups point of view and significantly extend the results of the previous section using this algebraic language. In particular, it allows us to compute the full exterior algebra and its cohomology as well as the full moduli of flat connections in the gauge theory picture that underlies the model. Note that our results here should not be confused with the question of existence or not of a spectral triple for a given differential calculus on a finite group, e.g. as in [13] and elsewhere (we are interested in a particular lattice Dirac operator $\partial\!\!\!/$).

We will use more algebraic notation. Thus, we work with the universal exterior algebra $\Omega_{\mathrm{univ}}(\mathcal{A})$ explicitly as an algebra with a finite number of left-invariant 1-forms as generators. We then exhibit $\Omega_{\partial\!\!\!/}(\mathcal{A})$ not as matrices but as a quotient $\Omega(\mathcal{A})$ of the universal one by relations among the generators (keeping the same names for the generators in the quotient). For ease of reference, the resulting dictionary with the concrete matrices in the previous section will be

$$\omega_x = \pi(e_x), \qquad \omega_y = \pi(e_y), \qquad H = \pi(\alpha), \qquad C = \pi(F), \qquad \varphi = \pi(\Phi) \tag{50}$$

for abstract forms e_x, e_y, α, Φ and F in $\Omega(A)$.

3.1. Exterior algebra and cohomology

A differential calculus from the quantum groups point of view means any $\mathcal{A}$–$\mathcal{A}$ bimodule $\Omega^1(\mathcal{A})$ and a map $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ obeying the Leibniz rule. When $\mathcal{A}$ is a Hopf algebra we demand further that $\Omega^1(\mathcal{A})$ is bicovariant [4]. Just as a topological space can admit more than one differential structure, one has to classify the possible $\Omega^1(\mathcal{A})$. From results in [4] it is immediate for the case of $\mathcal{A} = \mathbb{C}[G]$, the functions on a finite group, that the possible bicovariant calculi are in correspondence with subsets

$$\mathcal{C} \subset G, \qquad e \notin \mathcal{C}, \tag{51}$$

where e is the group identity. The elements of $\mathcal{C}$ label the 'basic 1-forms' $\{e_a\}$ of the corresponding $\Omega^1_{\mathcal{C}}(\mathcal{A})$ and any other 1-form is a unique linear combination of these with coefficients from $\mathcal{A}$. The commutation rules and general form of d in this construction are

$$e_a f = R_a(f) e_a, \qquad \mathrm{d}f = \sum_{a \in \mathcal{C}} (\partial_a f) e_a, \qquad \partial_a = R_a - \mathrm{id}. \tag{52}$$

One of the nice features of this construction is that it does not require the group to be Abelian, i.e. extends to non-Abelian or 'curved' lattices.

Also, these calculi are all quotients of the universal $\Omega^1_{\mathrm{univ}}(\mathcal{A})$ which can either be defined 'symbolically' as in the previous section or very explicitly as the elements of $\mathcal{A} \otimes \mathcal{A}$ whose product is zero. Here $\mathrm{d}f = 1 \otimes f - f \otimes 1$. For functions on a finite set G, we take for $\mathcal{A}$ a basis of δ-functions and hence

$$\Omega^1_{\mathrm{univ}}(\mathcal{A}) = \{\delta_g \otimes \delta_h = \delta_g \, \mathrm{d}\delta_h | g \neq h,\, g, h \in G\}. \tag{53}$$

The quotient to our chosen $\Omega^1_{\mathcal{C}}(\mathcal{A})$ means to set to zero all such elements except for those for which $(g, h) \in E$, some subset of allowed directions. In the group case this subset E

is defined in a translation-invariant manner from $\mathcal{C}$, namely as pairs (g, h) for which the difference (in the additive case) lives in $\mathcal{C}$.

In our case we choose the subset

$$\mathcal{C} = \{x, y\}, \quad x = (1, 0), \quad y = (0, 1), \tag{54}$$

so every 1-form is uniquely of the form

$$\alpha = \alpha_x e_x + \alpha_y e_y, \quad \alpha_x, \alpha_y \in \mathcal{A}. \tag{55}$$

The basic 1-forms can be written explicitly as

$$e_x = \sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} \delta_g \, \mathrm{d}\delta_{g+x}, \qquad e_y = \sum_{g\in\mathbb{Z}_2\times\mathbb{Z}_2} \delta_g \, \mathrm{d}\delta_{g+y}. \tag{56}$$

This is a full description of $\Omega^1_{\mathcal{C}}(\mathcal{A})$ as defined by the choice of $\mathcal{C}$ above. Clearly, we have the same answer as in Section 2 where we postulated an operator $\partial\!\!\!/$ and derived $\Omega^1_{\partial\!\!\!/}(\mathcal{A})$, i.e.

$$\pi : \Omega^1_{\mathcal{C}}(\mathcal{A}) \cong \Omega^1_{\partial\!\!\!/}(\mathcal{A}). \tag{57}$$

Actually this is a well-known general feature; for any linearly independent $\{\gamma^a\}$ and $\partial\!\!\!/ = \partial_a \gamma^a$, we will have the same agreement between the Connes and the quantum groups approach up to degree 1, by construction. We will work with this $\Omega^1_{\mathcal{C}}(\mathcal{A})$ and no longer write the $\mathcal{C}$ explicitly.

Next we consider higher degree forms. For any first-order calculus $\Omega^1(\mathcal{A})$ there is a 'linear prolongation' where we impose only the relations in higher forms inherited from those at degree 1 and $\mathrm{d}^2 = 0$. The latter in our case means

$$\begin{aligned} 0 = \mathrm{d}(\partial_x(f)e_x + \partial_y(f)e_y) = &-2\partial_x(f)e_x^2 - 2\partial_y(f)e_y + \partial_x\partial_y(f)(e_x e_y + e_y e_x) \\ &+ \partial_x(f)\,\mathrm{d}e_x + \partial_y(f)\,\mathrm{d}e_x, \end{aligned}$$

and choosing $f = \delta_{00} + \delta_{01} - \delta_{10} - \delta_{11}$ which obeys $\partial_x f = -2f$ and $\partial_y f = 0$, and a similar function for the roles of x, y interchanged, one finds

$$\mathrm{d}e_x = 2e_x^2, \qquad \mathrm{d}e_y = 2e_y^2, \quad e_x e_y = -e_y e_x. \tag{58}$$

The last of these follows from putting the first two into the $\mathrm{d}^2 = 0$ equation and then choosing a function with $\partial_x\partial_y f \neq 0$. Beyond this linear prolongation exterior algebra, we are free in the constructive approach to impose further relations in higher degrees. One general construction exists due to Woronowicz [4] and for an Abelian group as in our case it would simply imply that $e_x^2 = e_y^2 = 0$. We do not do this but instead impose the relation coming out of the Connes machinery in Section 2, namely

$$e_x^2 = e_y^2 \tag{59}$$

in the exterior algebra. Here the Connes approach and the Woronowicz approach for higher differentials diverge and we choose the former. Then $\Omega^2(\mathcal{A})$ is two-dimensional over $\mathcal{A}$, being spanned by $e_x e_y, e_y^2$. Choosing representatives in the universal exterior algebra for these, our explicit calculations (30) in the previous section show that their images under π are linearly independent, hence

$$\pi : \Omega^2(\mathcal{A}) \cong \Omega^2_{\not\partial}(\mathcal{A}) \tag{60}$$

when constructed in this way.

Next we take the 'quadratic prolongation' of this Ω^1, Ω^2 to degree 3 and higher, i.e. impose no further relations than the quadratic ones (58) and (59) already imposed and whatever is implied by these.

Proposition 3.1. *The quadratic exterior algebra $\Omega(\mathcal{A})$ generated by e_x, e_y with relations $e_x^2 = e_y^2$ and $\{e_x, e_y\} = 0$ is isomorphic to $\Omega_{\not\partial}(\mathcal{A})$. Moreover, there is a generating 1-form*

$$\theta = e_x + e_y, \qquad \mathrm{d}\alpha = \{\theta, \alpha]$$

for all forms α, where we use commutator on even degree and anticommutator on odd degree and $\pi(\theta) = \not\partial_G$ as a matrix.

Proof. First we compute what this quadratic exterior algebra looks like. We then compare it with Connes construction and check the isomorphism. The remark about θ is then an immediate corollary since it is a general feature of the linear prolongation of $\Omega^1(\mathcal{A})$ (where we have seen that e_x, e_y anticommute) and hence holds in the quadratic exterior algebra quotient (as well as in the Woronowicz exterior algebra where it is would be well known). To compute the quadratic exterior algebra we note that

$$\mathrm{d}(e_x e_y + e_y e_x) = 2e_x^2 e_y - e_x 2e_y^2 + 2e_y^2 e_x - e_y 2e_x^2 = 0, \qquad \mathrm{d}(e_x^2 - e_y^2) = 0$$

automatically, hence there are no implied relations in degree 3 or higher coming from these. In that case, we have only the relations (59) and the anticommutativity relations. From this it is easy to see that

$$\Omega^p(\mathcal{A}) = \mathcal{A}\langle e_x e_y^{p-1}, e_y^p \rangle, \quad p \geq 1 \tag{61}$$

is two-dimensional over $\mathcal{A} = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$. By comparison we recall Connes definition

$$\Omega^p_{\not\partial}(\mathcal{A}) = \frac{\pi_p(\Omega^p_{\mathrm{univ}}(\mathcal{A}))}{\mathcal{J}_p}, \quad \mathcal{J}_p = \pi_p(\mathrm{d}\ker \pi_{p-1}),$$

where π_p denotes π in degree p of the universal calculus. The quadratic exterior algebra is at least as big as the Connes one since it uses only the relations already holding in the latter in degrees 1 and 2. Hence all that we really need to establish an isomorphism is to show that $\Omega^p_{\not\partial}$ has dimension 2 over $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$. In fact, it suffices to exhibit two elements of the universal exterior algebra with linearly independent images in $\pi_p(\Omega^p_{\mathrm{univ}}(\mathcal{A}))$ for each p, after which the result can be proven by induction. Indeed, knowing our result for $\Omega^{p-1}_{\not\partial}(\mathcal{A}) \cong \Omega^{p-1}(\mathcal{A})$, we know that the kernel of π_{p-1} is generated by the quadratic relations above. But d of these, by the computation above, lies again in the ideal generated by these relations, so $\mathcal{J}_p = 0$ and hence $\Omega^p_{\not\partial}(\mathcal{A}) = \pi_p(\Omega^p_{\mathrm{univ}}(\mathcal{A}))$.

There are many ways to come up with the required two elements of the universal exterior algebra in each degree p. The natural construction is a method that works very generally for any finite group (see below). Alternatively we can use the representatives implicit in the ad hoc computations in Section 2. Thus we lift e_x, e_y to the universal exterior algebra as elements

$$\tilde{e}_x = (1-2X)\,\mathrm{d}X, \qquad \tilde{e}_y = (1-2Y)\,\mathrm{d}Y,$$

where X, Y are some functions as in Section 2. Here

$$\partial_x X = 1-2X, \qquad (1-2X)^2 = 1, \qquad (\mathrm{d}X)(1-2X) = -(1-2X)\,\mathrm{d}X,$$
$$[\partial\!\!\!/, X] = (1-2X) R_x \otimes \gamma^x,$$

and similarly for Y. We also need $Z = \delta_{10} + \delta_{01} = X + Y - 2XY$ which obeys

$$(\mathrm{d}X)(1-2Y) = \mathrm{d}Z - (1-2X)\,\mathrm{d}Y, \qquad [\partial\!\!\!/, Z] = (1-2Z)(R_x \otimes \gamma^x + R_y \otimes \gamma^y).$$

From these facts it is not hard to compute

$$\tilde{e}_y^p = (1-2Y)^{[p]} (\mathrm{d}Y)^p (-1)^{p(p-1)/2}, \qquad \pi_p(\tilde{e}_y^p) = (R_y \otimes \gamma^y)^{[p]},$$

where $[p] = p \bmod 2$, and

$$\tilde{e}_x \tilde{e}_y^{p-1} = (1-2X)(\mathrm{d}X)(1-2Y)^{[p-1]} (\mathrm{d}Y)^{p-1} (-1)^{(p-1)(p-2)/2},$$
$$\pi_p(\tilde{e}_x \tilde{e}_y^{p-1}) = (R_x \otimes \gamma^x)(R_y \otimes \gamma^y)^{[p-1]}.$$

These are linearly independent for each p as required. □

A more explicit way to obtain this result, which makes clearer the quotienting from the universal calculus (and works similarly for any finite group G), is to note that the universal calculus on a Hopf algebra is automatically bicovariant and hence in the group case corresponds to some subset, namely $\mathcal{C}_{\mathrm{univ}} = G - \{e\}$. In our particular case it means a basic 1-form

$$e_{x+y} = \sum_g \delta_g \,\mathrm{d}\delta_{g+x+y} \tag{62}$$

in addition to e_x, e_y defined in the same way by (56), but now in $\Omega_{\mathrm{univ}}(\mathcal{A})$. The universal exterior algebra is the free algebra generated over $\mathcal{A}$ by these e_g for all $g \in G - \{e\}$. Now on any bicovariant calculus (using Hopf algebra methods) one has a Maurer–Cartan equation, which, for the universal calculus in our case, comes out as

$$\mathrm{d}e_x = 2e_x^2 + \{e_x, e_y\} + \{e_x, e_{x+y}\} - \{e_y, e_{x+y}\}, \tag{63}$$

$$\mathrm{d}e_{x+y} = 2e_{x+y}^2 + \{e_x, e_{x+y}\} + \{e_y, e_{x+y}\} - \{e_x, e_y\}, \tag{64}$$

and a similar equation for $\mathrm{d}e_y$. Similarly for any finite group.

With this description of $\Omega_{\mathrm{univ}}(\mathcal{A})$ the linear prolongation exterior algebra mentioned above is just given by setting to zero all the e_a except those in our conjugacy class. In our case, we project out $e_{x+y} = 0$ and this yields the Maurer–Cartan equation for our calculus and the additional anticommutation relation, i.e. (58) as the linear prolongation. Likewise the quadratic exterior algebra adds the additional relation $e_x^2 = e_y^2$. Note also that the $\tilde{e}_x = e_x + e_{x+y}$ and $\tilde{e}_y = e_y + e_{x+y}$ used above project onto the same 1-forms under the quotient as our generators e_x, e_y, but are not so natural from the point of view of the group structure.

For the products of 1-forms in the universal calculus we note that $\delta_g(\mathrm{d}\delta_{g+x})\delta_h = \delta_g\,\mathrm{d}(\delta_{g+x}\delta_h)$, etc. Hence it is immediate that

$$e_y^p = \sum_g \delta_g\,\mathrm{d}\delta_{g+y}\,\mathrm{d}\delta_g\,\mathrm{d}\delta_{g+y}\,\mathrm{d}\delta_g \cdots ,$$
$$e_x e_y^{p-1} = \sum_g \delta_g\,\mathrm{d}\delta_{g+x}\,\mathrm{d}\delta_{g+x+y}\,\mathrm{d}\delta_{g+x}\,\mathrm{d}\delta_{g+x+y} \cdots \tag{65}$$

(alternating until the total degree is p). We also need

$$[\partial\!\!\!/, \delta_g] = (\delta_{g+x} - \delta_g)R_x \otimes \gamma^x + (\delta_{g+y} - \delta_g)R_y \otimes \gamma^y. \tag{66}$$

When computing π of products of the e_x, e_y the δ_g to the front forces which of the four δ-functions in each $[\partial\!\!\!/, \delta_\cdot]$ can contribute. Let a, b, c, etc. be chosen from $\{x, y\}$. Then similar to the above, we have

$$e_a e_b e_c \cdots = \sum_g \delta_g\,\mathrm{d}\delta_{g+a}\,\mathrm{d}\delta_{g+a+b}\,\mathrm{d}\delta_{g+a+b+c} \cdots ,$$

$$\begin{aligned}\pi(e_a e_b e_c \cdots) &= \sum_g \delta_g (R_a \otimes \gamma^a)\,\mathrm{d}\delta_{g+a+b}\,\delta_{g+a+b+c} \cdots \\ &= (R_a \otimes \gamma^a) \sum_g \delta_{g+a}\,\mathrm{d}\delta_{g+a+b}\,\mathrm{d}\delta_{g+a+b+c} \cdots \\ &= (R_a \otimes \gamma^a)\pi(e_b e_c \cdots)\end{aligned}$$

after a change of variables. Hence, we find for this description of the universal calculus that

$$\pi(e_a e_b e_c \cdots) = R_a R_b R_c \cdots \otimes \gamma^a \gamma^b \gamma^c \cdots . \tag{67}$$

In fact, we see explicitly that $\pi(e_x) = R_x \otimes \gamma^x$, $\pi(e_y) = R_y \otimes \gamma^y$ and $\pi(e_{x+y}) = 0$ is an algebra homomorphism when extended by $\pi(fe_x) = \rho(f)\pi(e_x)$, etc. Again, this is a general construction for any finite group, conjugacy class and choice of linearly independent 'gamma-matrices'. The map

$$\pi : \Omega_{\mathrm{univ}}(\mathcal{A}) \to \Omega_{\partial\!\!\!/}(\mathcal{A}), \qquad \pi(e_a) = \begin{cases} R_a \otimes \gamma^a & \text{for } a \in \mathcal{C}, \\ 0 & \text{for } a \notin \mathcal{C} \cup \{e\}, \end{cases} \tag{68}$$

is an algebra homomorphism with $\partial\!\!\!/ = \sum_a \partial_a \otimes \gamma^a$. Its kernel depends on the relations among the gamma-matrices; their only homogeneous relations being quadratic in our case (and G being Abelian) is the reason that $\Omega_{\partial\!\!\!/}(\mathcal{A})$ is quadratic.

This completes our algebraic description $\Omega(\mathcal{A})$ of the exterior algebra $\Omega_{\partial\!\!\!/}(\mathcal{A})$ obtained by Connes construction for our choice of $\partial\!\!\!/$. The various nonzero dimensions of the exterior algebra of $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ are 1:2:2:2, etc. and there is no top form. This means that one should not expect a Hodge $*$ operator or Poincaré duality for this calculus.

Proposition 3.2. *The quantum de Rham cohomology of this differential calculus on* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *is*

$$H^0 = \mathbb{C} \cdot 1, \qquad H^1 = \mathbb{C} \cdot (e_x - e_y), \qquad H^p = \{0\}, \quad p \geq 2.$$

Proof. If f is a function and $\mathrm{d}f = 0$ it means $R_x(f) = f$ and $R_y(f) = f$ and hence f is a multiple of the constant identity function. Hence H^0 is spanned by 1. If a 1-form (55) is closed it means (using the Leibniz rule and d as above) that

$$0 = \mathrm{d}\alpha = (\partial_x \alpha_y - \partial_y \alpha_x) e_x e_y + (\partial_x \alpha_x + \partial_y \alpha_y + 2\alpha_x + 2\alpha_y) e_y^2.$$

We write a 2-form $\alpha_{xy} e_x e_y + \alpha_{yy} e_y^2$ as a vector

$$\begin{pmatrix} \alpha_{xy} \\ \alpha_{yy} \end{pmatrix},$$

and α as a vector

$$\begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}.$$

Then the operator d_1 which is d on 1-forms is an 8×8 matrix

$$\mathrm{d}_1 = \begin{pmatrix} \mathrm{id} - R_y & R_x - \mathrm{id} \\ \mathrm{id} + R_x & \mathrm{id} + R_y \end{pmatrix},$$

and its kernel is easily found to be four-dimensional. The exact forms in Ω^1 form a three-dimensional subspace of this kernel (since $\mathrm{d} : \Omega^0(\mathcal{A}) \to \Omega^1(\mathcal{A})$ has one-dimensional kernel given by constants). Hence H^1 is one-dimensional and easily seen to be represented by $e_x - e_y$. Also note that the image of d_1 is therefore four-dimensional also. For the general $\mathrm{d}_p : \Omega^p \to \Omega^{p+1}$, we note that

$$\mathrm{d}e_y^p = \begin{cases} 2e_y^{p+1} & \text{for } p \text{ odd}, \\ 0 & \text{for } p \text{ even}, \end{cases}$$

as one may easily prove by the graded Leibniz rule and induction. Then

$$\begin{aligned} \mathrm{d}(fe_x e_y^{p-1} + ge_y^p) = (\partial_x f + \partial_y g + 2f) e_y^{p+1} + g\, \mathrm{d}e_y^p \\ + (\partial_x g - \partial_y f) e_x e_y^p - fe_x\, \mathrm{d}e_y^{p-1} \end{aligned}$$

corresponds to the matrix

$$\mathrm{d}_p = \begin{pmatrix} (-\mathrm{id})^{p-1} - R_y & R_x - \mathrm{id} \\ \mathrm{id} + R_x & (-\mathrm{id})^{p-1} + R_y \end{pmatrix},$$

which has an order 2 periodicity. In particular,

$$\mathrm{d}_2 = \begin{pmatrix} -(\mathrm{id} + R_y) & R_x - \mathrm{id} \\ \mathrm{id} + R_x & -(\mathrm{id} - R_y) \end{pmatrix}.$$

The transpose of this matrix is easily seen to be conjugate under

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

to $-\mathrm{d}_1$. Hence the kernel of d_2 has the same dimension as the kernel of d_1, namely 4. Hence $H^2 = \{0\}$. Also the image of d_2 is therefore four-dimensional as is the kernel of d_3 (by periodicity) hence $H^3 = \{0\}$. The rest vanish by periodicity. □

Let us note as an aside that in the simpler Woronowicz calculus where we would set $e_x^2 = e_y^2 = 0$, the cohomology by a similar computation is more easily found to be $H^0 = \mathbb{C} \cdot 1$, $H^1 = \mathbb{C} \cdot e_x \oplus \mathbb{C} \cdot e_y$ and $H^2 = \mathbb{C} \cdot e_x e_y$ which has dimensions 1:2:1. This is because in this case the kernel of d on 1-forms is five-dimensional. The exterior algebra in this case also has the symmetric form with dimensions 1:2:1 over $\mathbb{C}[Z_2 \times \mathbb{Z}_2]$ but this calculus is not the one coming out of our Dirac operator using Connes prescription.

3.2. *Gauge theory*

Returning to our above differential calculus, we can also impose a $*$-structure with e_x, e_y Hermitian as for ω_x, ω_y in Section 2. Note that then

$$(\mathrm{d}f)^* = e_x(\partial_x f)^* + e_y(\partial_y f)^* = e_x\partial_x(f^*) + e_y\partial_y(f^*) = -\mathrm{d}f^* \tag{69}$$

using the definition of ∂ and the commutation relations (52). (This is not a property of the Hilbert space representation.) Thus the real cohomology is $H^0 = \mathbb{R}$, etc.

Given a differential calculus one is also free to do ‘gauge theory’ with connections $\alpha \in \Omega^1(\mathcal{A})$. This is obviously some kind of $U(1)$ gauge theory. It is worth noting that from a fully noncommutative geometrical point of view [14], it would be better called ‘U(0)’, with $\mathbb{C}$ the enveloping algebra of the zero Lie algebra (or the coordinate ring of a point). We assume that α is Hermitian, which means

$$R_x(\alpha_x^*) = \alpha_x, \qquad R_y(\alpha_y^*) = \alpha_y \tag{70}$$

in terms of its components. Gauge transformation is by $u \in U(\mathcal{A})$ as

$$\alpha^u = u\alpha u^{-1} + u\,\mathrm{d}u^{-1}. \tag{71}$$

In our case a unitary u essentially means a function on $\mathbb{Z}_2 \times \mathbb{Z}_2$ with values in the unit circle, $u = e^{i\phi}$ with ϕ real, hence the above is explicitly

$$\alpha_x \mapsto uR_x(u^{-1})\alpha_x + u\partial_x u^{-1} = e^{-i\partial_x\phi}\alpha_x + e^{-i\partial_x\phi} - 1, \tag{72}$$

and similarly for α_y. The gauge-covariant curvature $F(\alpha) = d\alpha + \alpha^2$ by a similar calculation to the above is

$$\begin{aligned} F(\alpha) = {} & (2\alpha_x + 2\alpha_y + \partial_x\alpha_x + \partial_y\alpha_y + \alpha_x R_x(\alpha_x) + \alpha_y R_y(\alpha_y))e_x^2 \\ & + (\partial_x\alpha_y - \partial_y\alpha_x + \alpha_x R_x(\alpha_y) - \alpha_y R_y(\alpha_x))e_x e_y. \end{aligned}$$

This is just the same result as in Section 2 except that it is obtained now by working in $\Omega(\mathcal{A})$ and its algebraic relations as above, not by explicit matrix calculations. Here the two coefficients are F_{xx} and F_{xy} say, and transform by conjugation of F, which means

$$F_{xx} \mapsto F_{xx}, \qquad F_{xy} \mapsto e^{-i\partial_{x+y}\phi} F_{xy}, \tag{73}$$

where $\partial_{x+y} = R_x R_y - \mathrm{id}$.

Proposition 3.3. *The moduli space of zero curvature gauge fields modulo gauge equivalence is a real circle*

$$\lambda^2 + \mu^2 = 2$$

modulo $\lambda \mapsto -\lambda$ *or* $\mu \mapsto -\mu$. *The corresponding gauge fields are*

$$\alpha = (\lambda - 1)e_x + (\mu - 1)e_y.$$

Proof. It is easy to see that these are solutions of the zero curvature equation, which we leave to the reader. We have to show that any solution is gauge equivalent to one of these. First, we change variables to

$$\Phi = \alpha + \theta, \qquad \Phi_a = \alpha_a + 1 \tag{74}$$

in which case the curvature and gauge transformation by u have the form

$$\begin{aligned} & F_{xx} = \Phi_x R_x \Phi_x + \Phi_y R_y \Phi_y - 2, \qquad F_{xy} = \Phi_x R_x \Phi_y - \Phi_y R_y \Phi_x, \\ & \Phi_x \mapsto uR_x(u^{-1})\Phi_x, \end{aligned} \tag{75}$$

and similarly for Φ_y. The $F = 0$ equation clearly becomes

$$\Phi_x R_x \Phi_x + \Phi_y R_y \Phi_y = 2, \Phi_x R_x \Phi_y = \Phi_y R_y \Phi_x. \tag{76}$$

The first of these implies that

$$\begin{aligned} & R_x(\Phi_x R_x \Phi_x) = (R_x \Phi_x)\Phi_x, \\ & R_y(\Phi_x R_x \Phi_x) = R_y(2 - \Phi_y R_y \Phi_y) = 2 - \Phi_y R_y \Phi_y = \Phi_x R_x \Phi_x, \end{aligned}$$

hence

$$\Phi_x R_x \Phi_x = \lambda^2, \qquad \Phi_y R_y \Phi_y = \mu^2, \quad \lambda^2 + \mu^2 = 2 \tag{77}$$

for some real constants λ, μ. Here the reality property of α translates as $\Phi_x^* = R_x \Phi_x$, etc. and hence $\Phi_x R_x \Phi_x = |\Phi_x|^2 \geq 0$, etc. For the moment, we assume that $\lambda, \mu \neq 0$ and consider the degenerate cases later. Next we write out the content of the other equation of (76) at the four points of $\mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\Phi_x(0,0)\Phi_y(1,0) = \Phi_y(0,0)\Phi_x(0,1), \qquad \Phi_x(1,0)\Phi_y(0,0) = \Phi_y(1,0)\Phi_x(1,1), \tag{78}$$

$$\Phi_x(0,1)\Phi_y(1,1) = \Phi_y(0,1)\Phi_x(0,0), \qquad \Phi_x(1,1)\Phi_y(0,1) = \Phi_y(1,1)\Phi_x(1,0). \tag{79}$$

In view of (77), most of these equations are redundant and we just have

$$\frac{\Phi_x(0,0)}{\Phi_x(0,1)} = \frac{\Phi_y(0,0)}{\Phi_y(1,0)}.$$

Let

$$u(0,0) = 1, \qquad u(0,1) = \frac{\mu}{\Phi_y(0,0)},$$
$$u(1,0) = \frac{\lambda}{\Phi_x(0,0)}, \qquad u(1,1) = \frac{\Phi_x(1,1)\mu}{\Phi_y(0,0)\lambda},$$

which is unitary (each component has modulus 1) in view of (77). Then using the above explicit equations and (77), one may verify that

$$\Phi_x = uR_x u^{-1}\lambda, \qquad \Phi_y = uR_y u^{-1}\mu$$

as required. In the special case where $\lambda = 0$ we have $\Phi_x = 0$ due to (77). We take

$$u(0,1) = u(1,1) = 1, \qquad u(0,0) = \frac{\Phi_y(0,0)}{\mu}, \qquad u(1,0) = \frac{\Phi_y(1,0)}{\mu},$$

and verify that $\Phi_y = uR_y u^{-1}\mu$ as required and that u is unitary. Similarly for $\mu = 0$. In these constructions we are free to choose $\lambda, \mu \geq 0$, for example, but are also free to choose them in other quadrants of the circle, which means that the different quadrants are all gauge equivalent to the positive one. Finally, we consider two gauge fields in our moduli space for positive λ, μ and λ', μ'. If related by a gauge transformation, we would need $(u(0,0)/u(0,1))\mu = \mu' = (u(0,1)/u(0,0))\mu$ by looking at $\Phi_u(0,0)$ and $\Phi_u(0,1)$. These imply that $\mu'^2 = \mu^2$ and hence $\mu = \mu'$. Similarly for $\lambda = \lambda'$ and the degenerate cases. Hence there is precisely one zero curvature gauge field up to equivalence for each parameter pair in the positive quadrant. That is, the moduli space is exactly the circle modulo the reflections $\lambda \mapsto -\lambda$, $\mu \mapsto -\mu$ (or exactly the quarter circle with positive values). □

We see that there is an entire circle of zero curvature gauge fields with the four quadrants gauge equivalent to each other. This is the 'geometry' of the discrete part of our model. In particular, the two opposite diameters $\alpha = 0$, and $\alpha = -2\theta$ (or $\Phi = \pm\theta$) are in fact gauge equivalent. Note also that for the Woronowicz choice with $e_x^2 = e_y^2 = 0$ the above proof would work in just the same way since (78) and (79) alone imply that $|\Phi_x|^2 = \lambda^2$, $|\Phi_y|^2 = \mu^2$ as in (77) but without the constraint that the parameters lie on a circle. In this case the moduli space of zero curvature gauge fields up to equivalence would be the entire plane modulo the two reflections (a 1/4 plane), i.e. does not have such a nontrivial topology as our case.

Finally, coming from the Connes' construction, we have an inner product particularly on forms. This plays the role of Hodge $*$ and integration against the top form rolled into one (even though the former does not appear separately). According to Section 2 it is

$$(f, g) = (fe_x, ge_x) = (fe_y, ge_y) = (fe_x e_y, ge_x e_y) = (fe_x^2, ge_x^2) = 2(f, g)_{l^2} \tag{80}$$

in terms of the usual l^2 inner product on functions f, g (and zero for other combinations of our basic forms). As explained in Section 2 this defines the gauge field action

$$\begin{aligned}\tfrac{1}{2}(F, F) &= \|F_{xx}\|^2 + \|F_{xy}\|^2\\ &= \||\Phi_x|^2 + |\Phi_y|^2 - 2\|^2 + \|R_x(\Phi_x^*\Phi_y) - R_y(\Phi_x\Phi_y^*)\|^2\end{aligned} \tag{81}$$

in terms of the usual l^2 norm. Clearly, the above $F = 0$ solutions form a circle of minima for this action whose origin is the point $\alpha = -\theta$ or $\Phi = 0$. The points on this circle are not gauge invariant, being equivalent to their reflections in other quadrants as well as defining a whole manifold of their further gauge transforms. According to Section 2 the center point of the circle is also an extremum, a local maximum and gauge invariant. In this way, the gauge field action resembles the 'Mexican hat' potential for a Higgs field if we view Φ as an adjoint Higgs field of some kind rather than as a connection as in our discrete geometry above.

4. Particle physics Lagrangians

In this section, the discrete gauge connections or Higgs's H are promoted to genuine fields, i.e. space–time dependent vectors. As already in classical quantum mechanics, this promotion is achieved by tensorizing with functions. Let us denote by $\mathcal{F}$ the algebra of (smooth, complex valued) functions over four-dimensional space–time M. Consider the algebra $\mathcal{A}_t := \mathcal{F} \otimes \mathcal{A}$. The group of unitaries of the tensor algebra $\mathcal{A}_t$ is the gauged version of the group of unitaries $U(\mathcal{A}) =: \mathcal{G}$ of the internal algebra $\mathcal{A}$, i.e. the group of functions from space–time into the group $\mathcal{G}$. Consider the representation $\rho_t := \underline{\cdot} \otimes \rho$ of the tensor algebra on the tensor product $\mathcal{H}_t := \mathcal{S} \otimes \mathcal{H}$, where $\mathcal{S}$ is the Hilbert space of square integrable spinors on which functions act by multiplication: $(f\underline{\cdot}\psi)(x) := f(x)\psi(x)$, $f \in \mathcal{F}$, $\psi \in \mathcal{S}$. The space–time points are labeled x and there should not be confusions with the discrete label $x \in \mathbb{Z}_2$. We denote the Dirac operator on the continuous space–time M by $\partial\!\!\!/_M$ and its chirality operator by γ^5. The definition of the tensor product of Dirac operators,

$$\partial\!\!\!/_t := \partial\!\!\!/_M \otimes 1_8 + \gamma^5 \otimes \partial\!\!\!/ \tag{82}$$

comes from noncommutative geometry. We now repeat the above construction for the infinite dimensional algebra $\mathcal{A}_t$ with representation ρ_t and Dirac operator $\partial\!\!\!/_t$. As already stated, for $\mathcal{A} = \mathbb{C}$, $\mathcal{H} = \mathbb{C}$, $\partial\!\!\!/ = 0$, the differential algebra $\Omega_{\partial\!\!\!/_t}(\mathcal{A}_t)$ is isomorphic to the de Rham algebra of differential forms $\Omega(M, \mathbb{C})$. For $\mathcal{A} = \mathbb{C}^2$, $\mathcal{H} = \mathbb{C}^2$, we obtain the two sheeted universe, one of the first examples [9] to exhibit spontaneous symmetry breaking. For general $\mathcal{A}$, using the notations of Schücker and Zylinski [10], a Hermitian 1-form

$$H_t \in \Omega^1_{\not{\partial}_t}(\mathcal{A}_t), \qquad H_t^* = H_t$$

contains two pieces, a Hermitian Higgs *field* $H \in \Omega^0(M, \Omega^1_{\not{\partial}}(\mathcal{A}))$ and a genuine gauge field $A \in \Omega^1(M, \mathrm{i}\rho(\mathfrak{g}))$ with values in i times the Lie algebra of the group of unitaries

$$\mathfrak{g} := \{X \in \mathcal{A}, X^* + X = 0\}, \tag{83}$$

represented on $\mathcal{H}$. The curvature of H_t

$$C_t := \mathrm{d}_t H_t + H_t^2 \in \Omega^2_{\not{\partial}_t}(\mathcal{A}_t) \tag{84}$$

contains three pieces

$$C_t = C + F - \mathrm{D}\varphi\gamma^5, \tag{85}$$

the ordinary, now space–time dependent curvature $C = \mathrm{d}H + H^2$, the field strength

$$F := \mathrm{d}_M A + \tfrac{1}{2}[A, A] \in \Omega^2(M, \rho(\mathfrak{g})), \tag{86}$$

and the covariant derivative of homogeneous scalar variable $\varphi := H + \not{\partial}_{\mathcal{G}}$,

$$\mathrm{D}\varphi = \mathrm{d}_M\varphi + [A\varphi - \varphi A] \in \Omega^1(M, \Omega^1_{\not{\partial}}(\mathcal{A})). \tag{87}$$

Note that the covariant derivative may be applied to φ thanks to its homogeneous transformation law, Eq. (46).

The definition of the Higgs potential in the infinite dimensional space $\mathcal{A}_t$

$$V_t(H_t) := (C_t, C_t) \tag{88}$$

requires a suitable regularization of the sum of eigenvalues over the space of spinors $\mathcal{S}$. Here we have to suppose space–time to be compact and Euclidean. Then, the regularization is achieved by the Dixmier trace [7] which allows an explicit computation of V_t. One of the key features in the Connes–Lott scheme is that V_t alone reproduces the complete bosonic action of a Yang–Mills–Higgs model. Indeed, it consists of three pieces, the Yang–Mills action, the covariant Klein–Gordon action and an integrated Higgs potential

$$V_t(A + H) = \int_M \mathrm{tr}(F^* * F) + \int_M \mathrm{tr}(\mathrm{D}\varphi^* * \mathrm{D}\varphi) + \int_M *V(H). \tag{89}$$

The natural appearance of both the kinetic term for the Higgs and its potential is the key feature of the approach. Recall that in particle phenomenology these two pieces are added to the Yang–Mills action opportunistically in order to reconcile a model with experiment. Here these two pieces are derived from geometry.

As the preliminary Higgs potential V_0, the (final) Higgs potential V is calculated from the finite dimensional triple $(\mathcal{A}, \mathcal{H}, \not{\partial})$,

$$V := V_0 - \mathrm{tr}[\alpha C^* \alpha C] = \mathrm{tr}[(C - \alpha C)^*(C - \alpha C)], \tag{90}$$

where the linear map

$$\alpha : \Omega^2_{\partial\!\!\!/}(\mathcal{A}) \to \rho(\mathcal{A}) + \pi(\mathrm{d}\ker\pi_1) \tag{91}$$

is determined by the two equations

$$\mathrm{tr}[R^*(C - \alpha C)] = 0 \quad \text{for all } R \in \rho(\mathcal{A}), \tag{92}$$

$$\mathrm{tr}[K^*\alpha C] = 0 \quad \text{for all } K \in \pi(\mathrm{d}\ker\pi_1). \tag{93}$$

All remaining traces are over the finite dimensional Hilbert space $\mathcal{H}$. We denote the Hodge star by $*\cdot$. It should not be confused with the involution $\cdot^*$. Note the 'wrong' relative sign of the third term in Eq. (89). The sign is in fact correct for an Euclidean space–time.

A similar feature holds in the fermionic sector, where the completely covariant action $\psi^*(\partial\!\!\!/_t + H_t)\psi$ reproduces the complete fermionic action of a Yang–Mills–Higgs model. We denote by

$$\begin{aligned} &\psi = \psi_R + \psi_L \in \mathcal{H}_t = \mathcal{S} \otimes (\mathcal{H}_R \oplus \mathcal{H}_L),\\ &\psi_L := \tfrac{1}{2}(1-\gamma^3)\psi, \quad \psi_R := \tfrac{1}{2}(1+\gamma^3)\psi, \end{aligned} \tag{94}$$

the multiplets of chiral spinors and by ψ^* the dual of ψ with respect to the scalar product of the concerned Hilbert space. We set

$$\partial\!\!\!/_{\mathcal{G}} = \mathcal{M}^* \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{M} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{95}$$

$\mathcal{M}$ will turn out to be the fermionic mass matrix. Similarly, we set

$$H =: \tilde{h}^* \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \tilde{h} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \Omega^1_{\partial\!\!\!/}(\mathcal{A}), \tag{96}$$

$$\varphi = H + \partial\!\!\!/_{\mathcal{G}} =: \tilde{\varphi}^* \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \tilde{\varphi} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \Omega^1_{\partial\!\!\!/}(\mathcal{A}). \tag{97}$$

Then

$$\begin{aligned} \psi^*(\partial\!\!\!/_t + H_t)\psi &= \int_M *\psi^*(\partial\!\!\!/_M + \gamma(A))\psi + \int_M *(\psi_L^*\tilde{h}\gamma^5\psi_R + \psi_R^*\tilde{h}^*\gamma^5\psi_L)\\ &\quad + \int_M *(\psi_L^*\mathcal{M}\gamma^5\psi_R + \psi_R^*\mathcal{M}^*\gamma^5\psi_L)\\ &= \int_M *\psi^*(\partial\!\!\!/_M + \gamma(A))\psi + \int_M *(\psi_L^*\tilde{\varphi}\gamma^5\psi_R + \psi_R^*\tilde{\varphi}^*\gamma^5\psi_L) \end{aligned} \tag{98}$$

containing the ordinary Dirac action and the Yukawa couplings. Note the unusual appearance of γ^5 in the fermionic action (98). Just as the wrong signs in the bosonic action (89), these γ^5 are proper to the Euclidean signature and disappear in the Minkowski signature. For details, see the first reference of [1, Example 2], and [15, Section 6.9].

In our lattice model the junk $\pi(\mathrm{d}\ker\pi_1)$ is zero and solving Eqs. (92) and (93) is easy

$$C - \alpha C = \rho(\varphi_x R_x \varphi_y - \varphi_y R_y \varphi_x)\,\omega_x\omega_y, \tag{99}$$

implying that upon tensorizing with continuous space–time the Higgs potential,

$$\begin{aligned} V = 2\{&|\varphi_x(0,0)\varphi_y(1,1)^* - \varphi_x(1,1)^*\varphi_y(0,0)|^2 \\ &+ |\varphi_x(0,0)\varphi_y(0,0)^* - \varphi_x(1,1)^*\varphi_y(1,1)|^2\}, \end{aligned} \tag{100}$$

loses its precious property of spontaneous symmetry breaking. We only know of very few examples where the spontaneous symmetry breaking is lost after tensorizing, the first example being the Connes–Lott model of electro-weak forces with one generation of leptons [1]. Details can be found in [15, Section 4.6].

5. Discrete diffeomorphisms and spectral action

Let us summarize Connes' strategy up to this point. He reformulates Riemannian geometry algebraically in terms of spectral triples $(\mathcal{A}, \mathcal{H}, \partial\!\!\!/)$. This reformulation is general enough to never use the commutativity of the algebra $\mathcal{A}$ of functions. It is special enough to include generalizations of differential forms, exterior multiplication and derivative and the combination of Hodge star and integration needed to define a Yang–Mills action. On a finite dimensional spectral triple, such a Yang–Mills action looks generically like a Higgs potential and breaks the group of unitaries in $\mathcal{A}$ spontaneously. Tensorizing the finite dimensional spectral triple with the infinite dimensional, commutative spectral triple of a Riemannian manifold, 'almost commutative geometry', produces a complete Yang–Mills–Higgs model. In this setting of almost commutative geometry, the Higgs scalar is reduced to a pseudo-force of the Yang–Mills force. This situation is perfectly analogous to Minkowskian geometry (special relativity) reducing the magnetic force to a pseudo-force of the electric force: take an electric charge at rest, $\vec{B} = 0$, and change coordinates to a frame moving with constant velocity. After this Lorentz boost, a magnetic field $\vec{B}$ appears. Every pseudo-force is attached to a coordinate transformation, another example being centrifugal and Coriolis forces attached to the transformation to the rotating frame. The Higgs scalar is attached to a gauge transformation which in noncommutative geometry is a generalized coordinate transformation.

With his fluctuating metric, Connes goes one step further [2]. His algebraic reformulation of Riemannian geometry of course contains a generalization of the Riemannian metric, the Dirac operator $\partial\!\!\!/$. This generalization is special enough to allow for an algebraic reformulation of general relativity in terms of the commutative spectral triple of a Riemannian manifold. The kinematical part of this algebraic reconstruction is the fluctuating metric, the dynamical part is the spectral action [8]. Repeating this algebraic construction for almost commutative spectral triples produces in addition to general relativity some very special Yang–Mills–Higgs models. In this almost commutative setting, therefore, these very special Yang–Mills–Higgs forces are reduced to pseudo-forces of gravity. The electromagnetic, weak and strong forces are among these very special Yang–Mills–Higgs forces.

The central tool to construct the fluctuating metric is the lift of the group of automorphisms and unitaries of $\mathcal{A}$ to the Hilbert space $\mathcal{H}$. For the commutative triple of a Riemannian manifold, the automorphisms are the diffeomorphisms of the manifold, *the general coordinate transformations*, and their image under the lift are the local spin transformations. The unitaries are gauged $U(1)$ transformations. In the presence of the real structure, they are all lifted to the identity. Let us compute the lift in our lattice example. The automorphism group of our algebra $\mathcal{A} = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ is

$$\mathrm{Aut}(\mathcal{A}) = S_4 \ni P, \tag{101}$$

the group of permutations of the four points. It is the discrete version of the diffeomorphism group. We disregard complex conjugation, that is we do not consider $\mathcal{A}$ to be real. The group of unitaries

$$U(\mathcal{A}) = U(1)^4 \ni u(x, y) \tag{102}$$

is the discrete version of Maxwell's gauge group. Simultaneously, it plays the role of the gauged Lorentz group. We need to map both groups to the group of automorphisms lifted to the Hilbert space $\mathcal{H}$,

$$\begin{aligned}\mathrm{Aut}_{\mathcal{H}}(\mathcal{A}) := \{&U : \mathcal{H} \to \mathcal{H}, UU^* = U^*U = 1, [U, \gamma_3] = 0,\\ &\forall f \in \mathcal{A} U\rho(f)U^{-1} = \rho(\tilde{f}); \exists \tilde{f} \in \mathcal{A}\}.\end{aligned} \tag{103}$$

As mentioned our example does not satisfy Connes' first-order condition. Anyway, we would have a hard time to choose the sign of the square of the real structure since this square is $+1$ in dimension zero, -1 in dimension two. Therefore, we do not introduce a real structure in the definition of the lifted automorphisms. Every lifted automorphism U projects down to an automorphism $P = p(U)$ with $P(f) = \tilde{f}$. In our example, we have

$$\mathrm{Aut}_{\mathcal{H}}(\mathcal{A}) = S_4 \ltimes (U(1)^4_L \times U(1)^4_R) \ni (P, u_L(x, y), u_R(x, y)). \tag{104}$$

Let us denote the lifting homomorphism by $(L, \ell) : \mathrm{Aut}(\mathcal{A}) \ltimes U(\mathcal{A}) \to \mathrm{Aut}_{\mathcal{H}}(\mathcal{A})$. It must satisfy $(p \circ (L, \ell))(P, u) = P$. Let us start with the automorphisms alone, $L(P) = (\beta(P), u_L(P), u_R(P))$. The most general solution is $\beta(P) = P$, $u_L(P) = \sigma_L(P)1_4$, $u_R(P) = \sigma_R(P)1_4$, where the two functions $\sigma_{L,R} : S_4 \to \mathbb{Z}_2$ are either identically one or the signature of the permutation, four possibilities. We have written an 1_4 to indicate that the unitaries are rigid, i.e. independent of x and y. As unitary 8×8 matrices the four possible lifts take the form

$$L(P) = P \otimes [\tfrac{1}{2}(\sigma_L + \sigma_R)1_2 + \tfrac{1}{2}(\sigma_L - \sigma_R)\gamma^3]. \tag{105}$$

They only induce trivial fluctuations of the metric

$$L(P)\partial\!\!\!/ L(P)^{-1} = \pm(\tilde{\partial}_x \otimes \gamma^x + \tilde{\partial}_y \otimes \gamma^y), \quad \tilde{\partial}_\cdot = P\partial_\cdot P^{-1}. \tag{106}$$

This is in sharp contrast to the continuous case where the lifted diffeomorphisms induce the general curved metric starting from the flat one. Fortunately, upon tensorizing with a continuous space–time we obtain a general internal Dirac operator that acquires the status of the fermionic mass matrix. In the almost commutative setting, we will also see the lift of the unitaries of our internal algebra $\mathcal{A} = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$.

The automorphisms of $\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A}$ close to the identity are diffeomorphisms of space–time $\phi \in \mathrm{Diff}(M)$. The group of unitaries $U(\mathcal{A}_t)$ is the gauged ${}^MU(1)^4$ whose elements are functions from M to $U(1)^4$ that we denoted by $u = (u(0,0), u(1,0), u(0,1), u(1,1))$ as before. The group of automorphisms lifted to the Hilbert space has as component connected to the identity

$$\mathrm{Aut}_{\mathcal{H}_t}(\mathcal{A}_t) = \mathrm{Diff}(M) \ltimes {}^M(\mathrm{Spin}(4) \times U(1)_L^4 \times U(1)_R^4) \ni (\phi, u_L, u_R). \tag{107}$$

The lift $L(\phi)$ is described explicitly in [16] and locally it induces the general curved Dirac operator on M by fluctuating the flat one

$$L(\phi)\partial\!\!\!/_{\mathrm{flat}} L(\phi)^{-1} = \mathrm{i}\, \mathrm{e}_a^{-1\mu}\gamma^a \left[\frac{\partial}{\partial x^\mu} + \frac{1}{4}\omega_{bc\mu}\gamma^b\gamma^c\right] = \partial\!\!\!/_M \tag{108}$$

with tetrad coefficients e^a_μ and their torsionless spin connection 1-form $\omega_{bc\mu}\,\mathrm{d}x^\mu$. Let us concentrate on lifting the unitaries: $\ell(u) = \rho_t(u)$, i.e. $u_L = u_R$ meets all requirements: ℓ is a group homomorphism, and for every unitary u in $U(\mathcal{A}_t)$, $\ell(u)$ is a unitary operator on $\mathcal{H}_t$, $\ell(u)$ commutes with $\gamma^5 \otimes \gamma^3$ and $p \circ \ell(u) = 1_{\mathcal{A}_t}$. We are ready to fluctuate the metric again

$$\begin{aligned}\ell(u)\partial\!\!\!/_t \ell(u)^{-1} =:& \ \partial\!\!\!/_{t\,\mathrm{fluct}}\\ &= \mathrm{i}\, \mathrm{e}_a^{-1\mu}\gamma^a \left[\frac{\partial}{\partial x^\mu} \otimes 1_8 + \frac{1}{4}\omega_{bc\mu}\gamma^b\gamma^c \otimes 1_8 - 1_4 \otimes \mathrm{i}\rho(A_\mu)\right]\\ &\quad + \gamma^5 \otimes [H + \partial\!\!\!/]\\ &= \mathrm{i}\, \mathrm{e}_a^{-1\mu}\gamma^a \left[\frac{\partial}{\partial x^\mu} \otimes 1_8 + \frac{1}{4}\omega_{bc\mu}\gamma^b\gamma^c \otimes 1_8 - 1_4 \otimes \mathrm{i}\rho(A_\mu)\right] + \gamma^5 \otimes \varphi\end{aligned} \tag{109}$$

with the Yang–Mills connection 1-form $\mathrm{i}A_\mu\,\mathrm{d}x^\mu = u\,\mathrm{d}u^{-1}$. As in the Connes–Lott scheme, the Higgs scalar appears as a connection 1-form with respect to the internal spectral triple, $H = \pi(u\,\mathrm{d}u^{-1})/\mathrm{i} =: \varphi - \partial\!\!\!/$. As before we expand $\varphi =: \rho(\varphi_x)\omega_x + \rho(\varphi_y)\omega_y$ with four, now space–time dependent complex coefficients, $\varphi_x(0,0) = \varphi_x(1,0)^*$, $\varphi_x(1,1) = \varphi_x(0,1)^*$, $\varphi_y(0,0) = \varphi_y(0,1)^*$, $\varphi_y(1,1) = \varphi_y(1,0)^*$. The kinematics is defined by a metric encoded in $\partial\!\!\!/_M$ or its tetrad coefficients, by a Yang–Mills potential, i.e. a 1-form A with values in i times the Lie algebra of $U(\mathcal{A})$ and by four complex Higgs scalars.

In general relativity, the dynamics of the metric is essentially fixed by a diffeomorphism invariant action functional. In the setting of spectral triples, there is a natural automorphism invariant action functional, the trace of the fluctuated Dirac operator, i.e. of the Dirac operator that is minimally coupled to the metric, to the Yang–Mills potential and to the Higgs scalars. Since the Dirac operator is self-adjoint and anticommutes with the chirality $\gamma^5 \otimes \gamma^3$, its spectrum is even and it is enough to compute the trace of its square. Being divergent, this trace is regularized by a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ of sufficiently fast decrease and the celebrated spectral action of Chamseddine and Connes [8] reads

$$S[g, A, \Phi] = \mathrm{tr}\, f\left(\frac{\partial\!\!\!/^2_{t\,\mathrm{fluct}}}{\Lambda}\right). \tag{110}$$

For convenience, we have put in a scale factor Λ carrying the dimension of the eigenvalues of the Dirac operator, say GeV. Asymptotically for large Λ, the spectral action reproduces the Einstein–Hilbert action and a complete Yang–Mills–Higgs action. In this limit the regularizing function f is universal in the sense that the spectral action only depends on its first three 'moments', $f_0 := \int_0^\infty tf(t)\,\mathrm{d}t$, $f_2 := \int_0^\infty f(t)\,\mathrm{d}t$ and $f_4 = f(0)$. In particular, its Higgs potential is

$$V = \lambda\,\mathrm{tr}_8(\varphi^*\varphi\varphi^*\varphi) - \frac{\mu^2}{2}\,\mathrm{tr}_8(\varphi^*\varphi), \quad \lambda = \frac{\pi}{f_4}, \quad \frac{\mu^2}{2} = \left(\frac{f_2}{f_4}\right)\Lambda^2. \tag{111}$$

A straightforward calculation gives

$$\begin{aligned} V = {} & 2\lambda\{[|\varphi_x(0,0)|^2 + |\varphi_y(0,0)|^2]^2 + [|\varphi_x(0,0)|^2 + |\varphi_y(1,1)|^2]^2 + [|\varphi_x(1,1)|^2 \\ & + |\varphi_y(0,0)|^2]^2 + [|\varphi_x(1,1)|^2 + |\varphi_y(1,1)|^2]^2 + 2|\varphi_x(0,0)\varphi_y(1,1)^* \\ & - \varphi_x(1,1)^*\varphi_y(0,0)|^2 + 2|\varphi_x(0,0)\varphi_y(0,0)^* - \varphi_x(1,1)^*\varphi_y(1,1)|^2\} \\ & - \mu^2\{\varphi_x(0,0)^*\varphi_x(0,0) + \varphi_x(1,1)^*\varphi_x(1,1) \\ & + \varphi_y(0,0)^*\varphi_y(0,0) + \varphi_y(1,1)^*\varphi_y(1,1)\}. \end{aligned} \tag{112}$$

As its brother from Section 2, Eq. (49), this potential has continuously degenerate minima, $\varphi_x(0,0) = \varphi_x(1,1) = \mu/(2\sqrt{\lambda})\sin\beta$, $\varphi_y(0,0) = \varphi_y(1,1) = \mu/(2\sqrt{\lambda})\cos\beta$. All minima break the gauged ${}^M U(1)^4$ spontaneously down to a single, rigid $U(1)$, except when β is an integer multiple of $\pi/2$. Then the little group is $U(1)^2$.

6. Concluding remarks

We conclude the paper with a brief outline, using again our quantum group methods, of what happens for other lattices

$$G = (\mathbb{Z}_m)^n. \tag{113}$$

Clearly, one might turn to these for better approximations of n-dimensional tori.

We take $\mathcal{A} = \mathbb{C}[(\mathbb{Z}_m)^n]$ of course and the usual n-dimensional γ-matrices γ^i, $i = 1, \ldots, n$. The calculus has the allowed directions which are the standard basis vectors $\mathcal{C} = \{\vec{x}_i | i = 1, \ldots, n\}$ of the lattice, where $\vec{x}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the element of $(\mathbb{Z}_m)^n$ with 1 in the ith place. Thus, $\Omega^1(\mathcal{A})$ is spanned by $\{e_i | i = 1, \ldots, n\}$, where $e_i =: e_{\vec{x}_i}$ is a shorthand. Likewise $\partial_i = R_{e_{\vec{x}_i}} - \mathrm{id}$ is the lattice differential in the ith direction in $(\mathbb{Z}_m)^n$. This description is necessarily isomorphic to the 1-forms in Connes construction for $\partial\!\!\!/ = \sum_i \partial_i \otimes \gamma^i$.

For the higher forms, we first compute the linear prolongation of $\Omega^1(\mathcal{A})$. Whatever the Connes $\Omega_{\partial\!\!\!/}(\mathcal{A})$ is, it must be a quotient of this. Using the method of Section 3.1, we start with the universal exterior algebra with generators $\{e_{\vec{g}} | \vec{g} \in (\mathbb{Z}_m)^n, \vec{g} \neq 0\}$. The linear prolongation consists of setting to zero all except the $\{e_i\}$. However, the Maurer–Cartan equations in the universal exterior algebra are

$$\mathrm{d}e_{\vec{g}} = \{\theta_{\mathrm{univ}}, e_{\vec{g}}\} - \sum_{\vec{b}+\vec{c}=\vec{g},\,\vec{b},\vec{c}\neq 0} e_{\vec{b}} e_{\vec{c}}, \quad \theta_{\mathrm{univ}} = \sum_{\vec{g}\neq 0} e_{\vec{g}}. \tag{114}$$

This is a special case of the Maurer–Cartan equations for any Hopf algebra and in any case easily verified from the standard form of the $e_{\vec{g}}$ in terms of δ-functions on G. Projecting out all but the $\{e_i\}$ gives

$$\mathrm{d}e_i = \{\theta, e_i\}, \quad \theta = \sum_i e_i, \tag{115}$$

$$0 = \sum_{\vec{x}_i + \vec{x}_j = \vec{g}} e_i e_j \quad \forall \vec{g} \in (\mathbb{Z}_m)^n, \quad \vec{g} \neq 0. \tag{116}$$

In all these equations, addition of vectors is mod m. If $m > 2$ Eq. (116) has two nonempty cases. When $\vec{g} = 2\vec{x}_i$ for some i, we have the equation

$$e_i^2 = 0, \tag{117}$$

and when $\vec{g} = \vec{x}_i + \vec{x}_j$ for some $i \neq j$, we have

$$\{e_i, e_j\} = 0. \tag{118}$$

Hence in this case the linear prolongation already coincides with the Woronowicz exterior algebra, which in turn is the 'trivial' one similar to that of $\mathbb{R}^n$. The Connes exterior algebra cannot have stronger relations than this and hence this is also $\Omega_{\not\partial}(\mathcal{A})$ in this case. In particular, $e_i^2 = 0$ eliminates all of the interesting features of our model such as the Higgs potential and spontaneous symmetry breaking. The model in effect resembles more like flat space.

On the other hand, $m = 2$ is precisely the case where $2\vec{x}_i = 0$ and is therefore not one of the possibilities for $\vec{g}$ in (116). Thus in this case the linear prolongation has only the relation $\{e_i, e_j\} = 0$ for $i \neq j$, in particular $e_i^2 \neq 0$ as for our $\mathbb{Z}_2 \times \mathbb{Z}_2$ case. We also have $\not\partial$ Hermitian and the same properties for the ∂_i as in the $n = 2$ case. In particular, we have the same features of the Higgs potential, etc. Finally, since $(R_i)^2 = \mathrm{id}$ as before, we have $\pi(e_i^2) = (R_i \otimes \gamma^i)^2 = 1$ and similar features for the higher forms. In summary, our $\mathbb{Z}_2 \times \mathbb{Z}_2$ model is typical of the general $(\mathbb{Z}_2)^n$ for $n \geq 2$.

Finally, we remark that the methods in this paper do apply to other finite groups just as well. For example, they could also be applied to a non-Abelian group or 'curved lattice' as in [3,6]. The first of these papers also proposes a general choice of γ-matrices (namely built from an irreducible representation of the finite group) and explicitly proposes a Dirac operator for the permutation group S_3 in this way. Development of that model along similar lines to that here could be an interesting topic for further work.

We also note [17] which was archived shortly before ours, in which a general class of Dirac operators on Abelian groups is proposed, although without any of the special features of our specific $\mathbb{Z}_2 \times \mathbb{Z}_2$ model.

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